

Fueter's theorem for monogenic functions in biaxial symmetric domains

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Abstract

In this paper we generalize the result on Fueter's theorem from [10] by Eelbode et al. to the case of monogenic functions in biaxially symmetric domains. To obtain this result, Eelbode et al. used representation theory methods but their result also follows from a direct calculus we established in our paper [21]. In this paper we first generalize [21] to the biaxial case and derive the main result from that.

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1 Introduction

Let \mathbb{R}_m be the real Clifford algebra generated by the standard basis $\{e_1, \dots, e_m\}$ of the Euclidean space \mathbb{R}^m (see [2, 16]). The multiplication in this associative algebra is determined by the relations: $e_j^2 = -1$, $e_j e_k + e_k e_j = 0$, $1 \leq j \neq k \leq m$. Any Clifford number $a \in \mathbb{R}_m$ may thus be written as

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{R},$$

where the basis elements $e_A = e_{j_1} \dots e_{j_k}$ are defined for every subset $A = \{j_1, \dots, j_k\}$ of $\{1, \dots, m\}$ with $j_1 < \dots < j_k$ (for $A = \emptyset$ one puts $e_\emptyset = 1$).

Observe that \mathbb{R}^{m+1} may be naturally embedded in \mathbb{R}_m by associating to any element $(X_0, X_1, \dots, X_m) \in \mathbb{R}^{m+1}$ the paravector $X_0 + \underline{X} = X_0 + \sum_{j=1}^m X_j e_j$. Furthermore, by the above multiplication rules it follows that $\underline{X}^2 = -|\underline{X}|^2 = -\sum_{j=1}^m X_j^2$.

The even and odd subspaces \mathbb{R}_m^+ , \mathbb{R}_m^- are defined as

$$\mathbb{R}_m^+ = \left\{ a \in \mathbb{R}_m : a = \sum_{|A| \text{ even}} a_A e_A \right\}, \quad \mathbb{R}_m^- = \left\{ a \in \mathbb{R}_m : a = \sum_{|A| \text{ odd}} a_A e_A \right\},$$

where $|A| = j_1 + \dots + j_k$. The subspace \mathbb{R}_m^+ is also a subalgebra and we have that

$$\mathbb{R}_m = \mathbb{R}_m^+ \oplus \mathbb{R}_m^-.$$

Consider the Dirac operator $\partial_{\underline{X}}$ in \mathbb{R}^m given by

$$\partial_{\underline{X}} = \sum_{j=1}^m e_j \partial_{X_j},$$

which provides a factorization of the Laplacian, i.e. $\partial_{\underline{X}}^2 = -\Delta_{\underline{X}} = -\sum_{j=1}^m \partial_{X_j}^2$. Functions in the kernel of $\partial_{\underline{X}}$ are known as monogenic functions (see [1, 7, 9, 12, 13]).

Definition 1. A function $F : \Omega \rightarrow \mathbb{R}_m$ defined and continuously differentiable in an open set $\Omega \subset \mathbb{R}^m$ is said to be (left) monogenic in Ω if $\partial_{\underline{X}} F(\underline{X}) = 0$, $\underline{X} \in \Omega$. In a similar way one defines monogenicity with respect to the generalized Cauchy-Riemann operator $\partial_{X_0} + \partial_{\underline{X}}$ in \mathbb{R}^{m+1} .

It is clear that monogenic functions are harmonic. Furthermore, for the particular case $m = 1$ the equation $(\partial_{X_0} + \partial_{\underline{X}})F(X_0, \underline{X}) = 0$ is nothing but the classical Cauchy-Riemann system for holomorphic functions. This is not the only connection existing between holomorphic and monogenic functions as the following result shows (see [27]).

Theorem 1 (Fueter's theorem). Let $w(z) = u(x, y) + iv(x, y)$ be a holomorphic function in the open subset Ξ of the upper half-plane and assume that $P_K(\underline{X})$ is a homogeneous monogenic polynomial of degree K in \mathbb{R}^m . If m is odd, then the function

$$(\partial_{X_0}^2 + \Delta_{\underline{X}})^{K + \frac{m-1}{2}} \left[\left(u(X_0, |\underline{X}|) + \frac{\underline{X}}{|\underline{X}|} v(X_0, |\underline{X}|) \right) P_K(\underline{X}) \right] \quad (1)$$

is monogenic in $\Omega = \{(X_0, \underline{X}) \in \mathbb{R}^{m+1} : (X_0, |\underline{X}|) \in \Xi\}$.

The idea of using holomorphic functions to construct monogenic functions was first presented by Fueter [11] in the setting of quaternionic analysis ($m = 3$, $K = 0$) and for that reason Theorem 1 bears his name. In 1957 Sce [25] extended Fueter's idea to Clifford analysis by proving the validity of the above result for the case $K = 0$, m odd. Forty years later Qian [22] showed that a similar result holds when m is even. In the last years several articles have been published on this topic (see e.g. [3, 4, 5, 6, 8, 10, 15, 19, 24]). For more information we refer the reader to the survey article [23].

Consider the biaxial decomposition $\mathbb{R}^m = \mathbb{R}^p \oplus \mathbb{R}^q$, $p + q = m$. In this way, for any $\underline{X} \in \mathbb{R}^m$ we may write

$$\underline{X} = \underline{x} + \underline{y},$$

where $\underline{x} = \sum_{j=1}^p x_j e_j$ and $\underline{y} = \sum_{j=1}^q x_{p+j} e_{p+j}$. We shall denote by \mathbb{R}_p and \mathbb{R}_q the real Clifford algebras constructed over \mathbb{R}^p and \mathbb{R}^q respectively, i.e.

$$\mathbb{R}_p = \text{Alg}_{\mathbb{R}}\{e_1, \dots, e_p\}, \quad \mathbb{R}_q = \text{Alg}_{\mathbb{R}}\{e_{p+1}, \dots, e_m\}.$$

In this paper we further investigate the following generalization of Fueter's theorem to the biaxial case (see [18, 20, 24]). We note that in this setting there is a slight change regarding the initial function w . Namely, w will be assumed to be antiholomorphic, i.e. a solution of $\partial_z w = 0$, where $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$.

Theorem 2. Let $w(\bar{z}) = u(x, y) + iv(x, y)$ be an antiholomorphic function in an open subset of $\{(x, y) \in \mathbb{R}^2 : x, y > 0\}$. Suppose that $P_k(\underline{x}) : \mathbb{R}^p \rightarrow \mathbb{R}_p$ and $P_\ell(\underline{y}) : \mathbb{R}^q \rightarrow \mathbb{R}_q$ are homogeneous monogenic polynomials. If p and q are odd, then the functions

$$\begin{aligned} \text{Ft}_{p,q}^+ [w(\bar{z}), P_k(\underline{x}), P_\ell(\underline{y})] (\underline{X}) &= \Delta_{\underline{X}}^{k+\ell+\frac{m-2}{2}} \left[\left(u(|\underline{x}|, |\underline{y}|) + \frac{\underline{x}\underline{y}}{|\underline{x}||\underline{y}|} v(|\underline{x}|, |\underline{y}|) \right) P_k(\underline{x}) P_\ell(\underline{y}) \right] \\ \text{Ft}_{p,q}^- [w(\bar{z}), P_k(\underline{x}), P_\ell(\underline{y})] (\underline{X}) &= \Delta_{\underline{X}}^{k+\ell+\frac{m-2}{2}} \left[\left(\frac{\underline{x}}{|\underline{x}|} u(|\underline{x}|, |\underline{y}|) + \frac{\underline{y}}{|\underline{y}|} v(|\underline{x}|, |\underline{y}|) \right) P_k(\underline{x}) P_\ell(\underline{y}) \right] \end{aligned}$$

are monogenic.

It is remarkable that Theorem 1 is still true if $P_K(\underline{X})$ is replaced by a homogeneous monogenic polynomial $P_K(X_0, \underline{X})$ in \mathbb{R}^{m+1} (see [21]), or if the monogenicity condition on $P_K(\underline{X})$ is dropped. The latter result was proved in [10] with the help of representation theory, but it can also be derived using the results obtained in [21].

Motivated by [10] and using similar methods as in [21], we prove in this paper that Theorem 2 also holds if $P_k(\underline{x})$ and $P_\ell(\underline{y})$ are assumed to be only homogeneous polynomials.

2 A higher order version of Theorem 2

The goal in this section is to generalize Theorem 2 to a larger class of initial functions. More precisely, we shall assume that $w(z, \bar{z}) = u(x, y) + iv(x, y)$ is a solution of the equation

$$\partial_z \Delta_{x,y}^\mu w(z, \bar{z}) = 0, \quad \Delta_{x,y} = \partial_x^2 + \partial_y^2, \quad \mu \in \mathbb{N}_0. \quad (2)$$

In particular, poly-antiholomorphic functions of order $\mu + 1$ (i.e. solutions of $\partial_z^{\mu+1} w(z, \bar{z}) = 0$) clearly satisfy equation (2).

It is possible to compute in explicit form the monogenic function produced by Theorem 1 using the differential operators

$$\left(x^{-1} \frac{d}{dx} \right)^n, \quad \left(\frac{d}{dx} x^{-1} \right)^n, \quad n \geq 0. \quad (3)$$

Namely, function (1) equals

$$(2K + m - 1)!! \left((R^{-1} \partial_R)^{K+\frac{m-1}{2}} u(X_0, R) + \frac{X}{R} (\partial_R R^{-1})^{K+\frac{m-1}{2}} v(X_0, R) \right) P_K(\underline{X}),$$

where $R = |\underline{X}|$ (see [18, 19]).

The differential operators in (3) possess interesting properties (see [8, 18, 19]) and in this paper we shall use the following.

Lemma 1. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a infinitely differentiable function, then

$$\begin{aligned} \text{(i)} \quad \frac{d^2}{dx^2} \left(x^{-1} \frac{d}{dx} \right)^n f(x) &= \left(x^{-1} \frac{d}{dx} \right)^n \frac{d^2}{dx^2} f(x) - 2n \left(x^{-1} \frac{d}{dx} \right)^{n+1} f(x), \\ \text{(ii)} \quad \frac{d^2}{dx^2} \left(\frac{d}{dx} x^{-1} \right)^n f(x) &= \left(\frac{d}{dx} x^{-1} \right)^n \frac{d^2}{dx^2} f(x) - 2n \left(\frac{d}{dx} x^{-1} \right)^{n+1} f(x), \end{aligned}$$

$$(iii) \quad \left(\frac{d}{dx} x^{-1} \right)^n \frac{d}{dx} f(x) = \frac{d}{dx} \left(x^{-1} \frac{d}{dx} \right)^n f(x),$$

$$(iv) \quad \left(x^{-1} \frac{d}{dx} \right)^n \frac{d}{dx} f(x) - \frac{d}{dx} \left(\frac{d}{dx} x^{-1} \right)^n f(x) = 2nx^{-1} \left(\frac{d}{dx} x^{-1} \right)^n f(x).$$

Due to the decomposition $\mathbb{R}^m = \mathbb{R}^p \oplus \mathbb{R}^q$ it is convenient to split $\partial_{\underline{X}}$ and $\Delta_{\underline{X}}$ as

$$\begin{aligned} \partial_{\underline{X}} &= \partial_{\underline{x}} + \partial_{\underline{y}} = \sum_{j=1}^p e_j \partial_{x_j} + \sum_{j=1}^q e_{p+j} \partial_{x_{p+j}}, \\ \Delta_{\underline{X}} &= \Delta_{\underline{x}} + \Delta_{\underline{y}} = \sum_{j=1}^p \partial_{x_j}^2 + \sum_{j=1}^q \partial_{x_{p+j}}^2. \end{aligned}$$

Furthermore, for any $\underline{x} \in \mathbb{R}^p$ and $\underline{y} \in \mathbb{R}^q$ we put

$$\begin{aligned} \underline{\omega} &= \underline{x}/r, \quad r = |\underline{x}|, \\ \underline{\nu} &= \underline{y}/\rho, \quad \rho = |\underline{y}|. \end{aligned}$$

In this section, like in Theorem 2, we assume that $P_k(\underline{x}) : \mathbb{R}^p \rightarrow \mathbb{R}_p$ and $P_\ell(\underline{y}) : \mathbb{R}^q \rightarrow \mathbb{R}_q$ are homogeneous monogenic polynomials. It is convenient to make a few observations about these polynomials.

Remark 1. First, note that $P_k(\underline{x})$ can be uniquely written in the form $P_k(\underline{x}) = P_k^+(\underline{x}) + P_k^-(\underline{x})$, where $P_k^+(\underline{x})$, $P_k^-(\underline{x})$ take values in \mathbb{R}_p^+ , \mathbb{R}_p^- respectively. Since $\partial_{\underline{x}} P_k^+(\underline{x}) \in \mathbb{R}_p^-$, $\partial_{\underline{x}} P_k^-(\underline{x}) \in \mathbb{R}_p^+$ for $\underline{x} \in \mathbb{R}^p$, one can conclude that $P_k(\underline{x})$ is monogenic if and only if both components $P_k^+(\underline{x})$ and $P_k^-(\underline{x})$ are monogenic. Of course, a similar remark holds for $P_\ell(\underline{y})$.

Let $\Delta_2 = \partial_r^2 + \partial_\rho^2$ be the two-dimensional Laplacian in the variables (r, ρ) and recall the definition of a multinomial coefficient

$$\binom{n}{j_1, j_2, \dots, j_s} = \frac{n!}{j_1! j_2! \cdots j_s!}.$$

Consider the function $D : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{Z}$ satisfying

$$\begin{aligned} D(0, 0) &= 1, \quad D(j_1, j_2) = D(j_1, 0)D(0, j_2), \quad j_1, j_2 \geq 1 \\ D(j, 0) &= \prod_{s=1}^j (2k + p - (2s - 1)), \quad D(0, j) = \prod_{s=1}^j (2\ell + q - (2s - 1)), \quad j \geq 1. \end{aligned}$$

Lemma 2. Suppose that $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ is an infinitely differentiable function in an open subset of $\{(x, y) \in \mathbb{R}^2 : x, y > 0\}$. Then for $n \in \mathbb{N}$ and $s_1, s_2 \in \{0, 1\}$ it holds that

$$\begin{aligned} \Delta_{\underline{X}}^n \left(h(r, \rho) \underline{\omega}^{s_1} \underline{\nu}^{s_2} P_k(\underline{x}) P_\ell(\underline{y}) \right) &= \\ \left(\sum_{\substack{j_1 + j_2 \leq n \\ j_1, j_2 \geq 0}} \binom{n}{j_1, j_2, n - j_1 - j_2} D(j_1, j_2) W_{j_1, j_2}^{s_1, s_2}(r, \rho) \right) \underline{\omega}^{s_1} \underline{\nu}^{s_2} P_k(\underline{x}) P_\ell(\underline{y}), \quad (4) \end{aligned}$$

where

$$\begin{aligned} W_{j_1, j_2}^{0,0}(r, \rho) &= (r^{-1} \partial_r)^{j_1} (\rho^{-1} \partial_\rho)^{j_2} \Delta_2^{n-j_1-j_2} h(r, \rho), \\ W_{j_1, j_2}^{1,0}(r, \rho) &= (\partial_r r^{-1})^{j_1} (\rho^{-1} \partial_\rho)^{j_2} \Delta_2^{n-j_1-j_2} h(r, \rho), \\ W_{j_1, j_2}^{0,1}(r, \rho) &= (r^{-1} \partial_r)^{j_1} (\partial_\rho \rho^{-1})^{j_2} \Delta_2^{n-j_1-j_2} h(r, \rho), \\ W_{j_1, j_2}^{1,1}(r, \rho) &= (\partial_r r^{-1})^{j_1} (\partial_\rho \rho^{-1})^{j_2} \Delta_2^{n-j_1-j_2} h(r, \rho). \end{aligned}$$

Proof. We shall prove the case $s_1 = s_2 = 0$ using induction. The other cases can be proved similarly. First, note that

$$\partial_{\underline{x}} h = \sum_{j=1}^p e_j \partial_{x_j} h = \sum_{j=1}^p e_j (\partial_r h) (\partial_{x_j} r) = \underline{\omega} \partial_r h$$

and hence

$$\begin{aligned} \Delta_{\underline{x}} h &= -\partial_{\underline{x}}^2 h = -\partial_{\underline{x}} (\underline{\omega} \partial_r h) = -\underline{\omega}^2 \partial_r^2 h - (\partial_{\underline{x}} \underline{\omega}) (\partial_r h) \\ &= \partial_r^2 h + \frac{p-1}{r} \partial_r h. \end{aligned}$$

From this equality and using Euler's theorem for homogeneous functions we obtain

$$\begin{aligned} \Delta_{\underline{x}} (h P_k) &= (\Delta_{\underline{x}} h) P_k + 2 \sum_{j=1}^p (\partial_{x_j} h) (\partial_{x_j} P_k) + h (\Delta_{\underline{x}} P_k) \\ &= \left(\partial_r^2 h + \frac{p-1}{r} \partial_r h \right) P_k + 2 \frac{\partial_r h}{r} \sum_{j=1}^p x_j \partial_{x_j} P_k = \left(\partial_r^2 h + \frac{2k+p-1}{r} \partial_r h \right) P_k. \end{aligned}$$

In a similar way one also get

$$\Delta_{\underline{y}} (h P_\ell) = \left(\partial_\rho^2 h + \frac{2\ell+q-1}{\rho} \partial_\rho h \right) P_\ell.$$

These equalities then yield

$$\Delta_{\underline{X}} (h P_k P_\ell) = \left(\Delta_2 h + \frac{2k+p-1}{r} \partial_r h + \frac{2\ell+q-1}{\rho} \partial_\rho h \right) P_k P_\ell. \quad (5)$$

It is clear that the assertion is true in the case $n = 1$. Assume now that the identity holds for some natural number n . We thus get

$$\begin{aligned} \Delta_{\underline{X}}^{n+1} (h P_k P_\ell) &= \sum_{\substack{j_1+j_2 \leq n \\ j_1, j_2 \geq 0}} \binom{n}{j_1, j_2, n-j_1-j_2} D(j_1, j_2) \\ &\quad \times \Delta_{\underline{X}} \left((r^{-1} \partial_r)^{j_1} (\rho^{-1} \partial_\rho)^{j_2} \Delta_2^{n-j_1-j_2} h P_k P_\ell \right). \end{aligned}$$

By statement (i) of Lemma 1 we obtain

$$\begin{aligned} \Delta_2 (r^{-1} \partial_r)^{j_1} (\rho^{-1} \partial_\rho)^{j_2} \Delta_2^{n-j_1-j_2} h &= (r^{-1} \partial_r)^{j_1} (\rho^{-1} \partial_\rho)^{j_2} \Delta_2^{n+1-j_1-j_2} h \\ &\quad - 2j_1 (r^{-1} \partial_r)^{j_1+1} (\rho^{-1} \partial_\rho)^{j_2} \Delta_2^{n-j_1-j_2} h - 2j_2 (r^{-1} \partial_r)^{j_1} (\rho^{-1} \partial_\rho)^{j_2+1} \Delta_2^{n-j_1-j_2} h. \end{aligned}$$

This equality and (5) imply that

$$\begin{aligned} D(j_1, j_2) \Delta_{\underline{X}} \left((r^{-1} \partial_r)^{j_1} (\rho^{-1} \partial_\rho)^{j_2} \Delta_2^{n-j_1-j_2} h P_k P_\ell \right) = \\ \left(D(j_1, j_2) (r^{-1} \partial_r)^{j_1} (\rho^{-1} \partial_\rho)^{j_2} \Delta_2^{n+1-j_1-j_2} h + D(j_1+1, j_2) (r^{-1} \partial_r)^{j_1+1} (\rho^{-1} \partial_\rho)^{j_2} \Delta_2^{n-j_1-j_2} h \right. \\ \left. + D(j_1, j_2+1) (r^{-1} \partial_r)^{j_1} (\rho^{-1} \partial_\rho)^{j_2+1} \Delta_2^{n-j_1-j_2} h \right) P_k P_\ell. \end{aligned}$$

Therefore

$$\Delta_{\underline{X}}^{n+1} (h P_k P_\ell) = (T_1 + T_2 + T_3) P_k P_\ell, \quad (6)$$

where

$$\begin{aligned} T_1 &= \sum_{\substack{j_1+j_2 \leq n \\ j_1, j_2 \geq 0}} \binom{n}{j_1, j_2, n-j_1-j_2} D(j_1, j_2) (r^{-1} \partial_r)^{j_1} (\rho^{-1} \partial_\rho)^{j_2} \Delta_2^{n+1-j_1-j_2} h, \\ T_2 &= \sum_{\substack{j_1+j_2 \leq n+1 \\ j_1 \geq 1, j_2 \geq 0}} \binom{n}{j_1-1, j_2, n+1-j_1-j_2} D(j_1, j_2) (r^{-1} \partial_r)^{j_1} (\rho^{-1} \partial_\rho)^{j_2} \Delta_2^{n+1-j_1-j_2} h, \\ T_3 &= \sum_{\substack{j_1+j_2 \leq n+1 \\ j_1 \geq 0, j_2 \geq 1}} \binom{n}{j_1, j_2-1, n+1-j_1-j_2} D(j_1, j_2) (r^{-1} \partial_r)^{j_1} (\rho^{-1} \partial_\rho)^{j_2} \Delta_2^{n+1-j_1-j_2} h. \end{aligned}$$

Observe that set $\{(j_1, j_2) : j_1 + j_2 \leq n+1, j_1, j_2 \geq 0\}$ can be expressed as the union of the disjoint sets $\{(0, 0)\}$, $\{(n+1, 0)\}$, $\{(0, n+1)\}$, $\{(j, 0) : 1 \leq j \leq n\}$, $\{(0, j) : 1 \leq j \leq n\}$, $\{(j, n+1-j) : 1 \leq j \leq n\}$ and $\{(j_1, j_2) : j_1 + j_2 \leq n, j_1, j_2 \geq 1\}$. Taking this into account it is easy to verify that (6) equals

$$\left(\sum_{\substack{j_1+j_2 \leq n+1 \\ j_1, j_2 \geq 0}} \binom{n+1}{j_1, j_2, n+1-j_1-j_2} D(j_1, j_2) (r^{-1} \partial_r)^{j_1} (\rho^{-1} \partial_\rho)^{j_2} \Delta_2^{n+1-j_1-j_2} h \right) P_k P_\ell.$$

Thus proving the assertion for $n+1$. \square

Remark 2. *Theorem 2 yields biaxial monogenic functions, i.e. monogenic functions of the form*

$$(A(r, \rho) + \underline{\omega} \underline{\nu} B(r, \rho)) P_k(\underline{x}) P_\ell(\underline{y})$$

or

$$(\underline{\omega} C(r, \rho) + \underline{\nu} D(r, \rho)) P_k(\underline{x}) P_\ell(\underline{y}),$$

where A, B, C, D are \mathbb{R} -valued continuously differentiable functions in \mathbb{R}^2 (see [14, 17, 26]). A direct computation shows that the pairs (A, B) and (C, D) satisfy the following Vekua-type systems

$$\begin{cases} \partial_r A + \partial_\rho B = -\frac{2\ell+q-1}{\rho} B \\ \partial_\rho A - \partial_r B = \frac{2k+p-1}{r} B, \end{cases} \quad \begin{cases} \partial_r C + \partial_\rho D = -\frac{2k+p-1}{r} C - \frac{2\ell+q-1}{\rho} D \\ \partial_\rho C - \partial_r D = 0. \end{cases}$$

We now come to our first main result that generalizes [21] to the biaxial case.

Theorem 3. *Suppose that $w(z, \bar{z}) = u(x, y) + iv(x, y)$ is a \mathbb{C} -valued function satisfying the equation (2) in the open set $\Xi \subset \{(x, y) \in \mathbb{R}^2 : x, y > 0\}$. If p and q are odd, then the functions*

$$\mathbf{Ft}_{p,q}^{\mu,+} [w(z, \bar{z}), P_k(\underline{x}), P_\ell(\underline{y})] (\underline{X}) = \Delta_{\underline{X}}^{\mu+k+\ell+\frac{m-2}{2}} \left[(u(r, \rho) + \underline{\omega} \underline{\nu} v(r, \rho)) P_k(\underline{x}) P_\ell(\underline{y}) \right],$$

$$\mathbf{Ft}_{p,q}^{\mu,-} [w(z, \bar{z}), P_k(\underline{x}), P_\ell(\underline{y})] (\underline{X}) = \Delta_{\underline{X}}^{\mu+k+\ell+\frac{m-2}{2}} \left[(\underline{\omega} u(r, \rho) + \underline{\nu} v(r, \rho)) P_k(\underline{x}) P_\ell(\underline{y}) \right]$$

are monogenic in $\Omega = \{\underline{X} = (\underline{x}, \underline{y}) \in \mathbb{R}^m : (r, \rho) \in \Xi\}$.

Proof. We use Lemma 2 to compute $\mathbf{Ft}_{p,q}^{\mu,+}$ in closed form. First, note that function w also satisfies the equation $\Delta_2^{\mu+1} w = 0$, thus

$$\Delta_2^{\mu+k+\ell+\frac{m-2}{2}-j_1-j_2} w = 0 \text{ for } j_1 + j_2 \leq k + \ell + (m-4)/2.$$

Since p and q are odd we also have that $D(j_1, j_2) = 0$ for $j_1 \geq k + (p+1)/2$ or $j_2 \geq \ell + (q+1)/2$. It follows that for $n = \mu + k + \ell + (m-2)/2$ the only term in (4) which does not vanish corresponds to the case $j_1 = k + (p-1)/2$, $j_2 = \ell + (q-1)/2$. Therefore

$$\begin{aligned} \mathbf{Ft}_{p,q}^{\mu,+} [w(z, \bar{z}), P_k(\underline{x}), P_\ell(\underline{y})] (\underline{X}) &= (2k+p-1)!!(2\ell+q-1)!! \\ &\times \binom{\mu+k+\ell+\frac{m-2}{2}}{k+\frac{p-1}{2}, \ell+\frac{q-1}{2}, \mu} (A(r, \rho) + \underline{\omega} \underline{\nu} B(r, \rho)) P_k(\underline{x}) P_\ell(\underline{y}), \end{aligned}$$

with

$$A = (r^{-1} \partial_r)^{k+\frac{p-1}{2}} (\rho^{-1} \partial_\rho)^{\ell+\frac{q-1}{2}} \Delta_2^\mu u, \quad B = (\partial_r r^{-1})^{k+\frac{p-1}{2}} (\partial_\rho \rho^{-1})^{\ell+\frac{q-1}{2}} \Delta_2^\mu v.$$

It thus remains to prove that (A, B) fulfills the first system of Remark 2. Using statement (iii) of Lemma 1 and the fact that w satisfies (2) we obtain

$$\partial_r A = (\partial_r r^{-1})^{k+\frac{p-1}{2}} (\rho^{-1} \partial_\rho)^{\ell+\frac{q-1}{2}} \partial_r \Delta_2^\mu u = -(\partial_r r^{-1})^{k+\frac{p-1}{2}} (\rho^{-1} \partial_\rho)^{\ell+\frac{q-1}{2}} \partial_\rho \Delta_2^\mu v.$$

Hence we get

$$\begin{aligned} \partial_r A + \partial_\rho B &= -(\partial_r r^{-1})^{k+\frac{p-1}{2}} \left((\rho^{-1} \partial_\rho)^{\ell+\frac{q-1}{2}} \partial_\rho \Delta_2^\mu v - \partial_\rho (\partial_\rho \rho^{-1})^{\ell+\frac{q-1}{2}} \Delta_2^\mu v \right) \\ &= -\frac{2\ell+q-1}{\rho} (\partial_r r^{-1})^{k+\frac{p-1}{2}} (\partial_\rho \rho^{-1})^{\ell+\frac{q-1}{2}} \Delta_2^\mu v, \end{aligned}$$

where we have also used statement (iv) of Lemma 1. In a similar fashion, it can be shown that

$$\partial_\rho A - \partial_r B = \frac{2k+p-1}{r} (\partial_r r^{-1})^{k+\frac{p-1}{2}} (\partial_\rho \rho^{-1})^{\ell+\frac{q-1}{2}} \Delta_2^\mu v.$$

The proof of $\mathbf{Ft}_{p,q}^{\mu,-}$ goes along the same lines as that of $\mathbf{Ft}_{p,q}^{\mu,+}$. Indeed, it follows from Lemma 2 that

$$\begin{aligned} \mathbf{Ft}_{p,q}^{\mu,-} [w(z, \bar{z}), P_k(\underline{x}), P_\ell(\underline{y})] (\underline{X}) &= (2k+p-1)!!(2\ell+q-1)!! \\ &\times \binom{\mu+k+\ell+\frac{m-2}{2}}{k+\frac{p-1}{2}, \ell+\frac{q-1}{2}, \mu} (\underline{\omega} C(r, \rho) + \underline{\nu} D(r, \rho)) P_k(\underline{x}) P_\ell(\underline{y}), \end{aligned}$$

with

$$C = (\partial_r r^{-1})^{k+\frac{p-1}{2}} (\rho^{-1} \partial_\rho)^{\ell+\frac{q-1}{2}} \Delta_2^\mu u, \quad D = (r^{-1} \partial_r)^{k+\frac{p-1}{2}} (\partial_\rho \rho^{-1})^{\ell+\frac{q-1}{2}} \Delta_2^\mu v.$$

One can check, using statements (iii) and (iv) of Lemma 1 as well as equation (2), that (C, D) satisfies the second system of Remark 2. \square

3 Fueter's theorem with general homogeneous factors

We arrive at the third and last section of the paper where we shall prove our main result. In the proof we use Theorem 3 and the well-known Fischer decomposition (see [9]):

Theorem 4 (Fischer decomposition). *Every homogeneous polynomial $H_K(\underline{X})$ of degree K in \mathbb{R}^m admits the following decomposition*

$$H_K(\underline{X}) = \sum_{n=0}^K \underline{X}^n P_{K-n}(\underline{X}),$$

where each $P_{K-n}(\underline{X})$ is a homogeneous monogenic polynomial.

Theorem 5. *Let $w(\bar{z}) = u(x, y) + iv(x, y)$ be an antiholomorphic function in the open set $\Xi \subset \{(x, y) \in \mathbb{R}^2 : x, y > 0\}$. Suppose that $H_k(\underline{x}) : \mathbb{R}^p \rightarrow \mathbb{R}_p$ and $H_\ell(\underline{y}) : \mathbb{R}^q \rightarrow \mathbb{R}_q$ are homogeneous polynomials. If p and q are odd, then the functions*

$$\text{Ft}_{p,q}^+ [w(\bar{z}), H_k(\underline{x}), H_\ell(\underline{y})] (\underline{X}) \quad \text{and} \quad \text{Ft}_{p,q}^- [w(\bar{z}), H_k(\underline{x}), H_\ell(\underline{y})] (\underline{X})$$

are monogenic in $\Omega = \{\underline{X} = (\underline{x}, \underline{y}) \in \mathbb{R}^m : (r, \rho) \in \Xi\}$.

Proof. We only prove the statement for function $\text{Ft}_{p,q}^+$ since the proof for $\text{Ft}_{p,q}^-$ is similar. Note that the Fischer decomposition ensures the existence of homogeneous monogenic polynomials $P_{k-n_1}(\underline{x})$ and $P_{\ell-n_2}(\underline{y})$ such that

$$H_k(\underline{x}) H_\ell(\underline{y}) = \sum_{n_1=0}^k \sum_{n_2=0}^\ell \underline{x}^{n_1} P_{k-n_1}(\underline{x}) \underline{y}^{n_2} P_{\ell-n_2}(\underline{y}).$$

This gives

$$\begin{aligned} & \text{Ft}_{p,q}^+ [w(\bar{z}), H_k(\underline{x}), H_\ell(\underline{y})] (\underline{X}) \\ &= \sum_{n_1=0}^k \sum_{n_2=0}^\ell \Delta_{\underline{X}}^{k+\ell+\frac{m-2}{2}} \left[(u(r, \rho) + \underline{\omega} \underline{\nu} v(r, \rho)) \underline{x}^{n_1} P_{k-n_1}(\underline{x}) \underline{y}^{n_2} P_{\ell-n_2}(\underline{y}) \right]. \end{aligned}$$

It will thus be sufficient to prove the monogenicity of each term in the previous sum. On account of Remark 1 we may assume without loss of generality that $P_{k-n_1}(\underline{x})$ takes values in \mathbb{R}_p^+ and hence

$$\underline{x}^{n_1} P_{k-n_1}(\underline{x}) \underline{y}^{n_2} P_{\ell-n_2}(\underline{y}) = \underline{x}^{n_1} \underline{y}^{n_2} P_{k-n_1}(\underline{x}) P_{\ell-n_2}(\underline{y}).$$

It is easy to verify that if $n_1 + n_2$ is even, then

$$\begin{aligned} & \Delta_{\underline{X}}^{k+\ell+\frac{m-2}{2}} \left[(u(r, \rho) + \underline{\omega} \underline{\nu} v(r, \rho)) \underline{x}^{n_1} \underline{y}^{n_2} P_{k-n_1}(\underline{x}) P_{\ell-n_2}(\underline{y}) \right] \\ &= \text{Ft}_{p,q}^{n_1+n_2,+} [w(\bar{z}) h^+(x, y), P_{k-n_1}(\underline{x}), P_{\ell-n_2}(\underline{y})] (\underline{X}), \end{aligned}$$

$$\text{where } h^+(x, y) = \begin{cases} (-1)^{\frac{n_1+n_2}{2}} x^{n_1} y^{n_2} & \text{for } n_1, n_2 \text{ even} \\ (-1)^{\frac{n_1+n_2-2}{2}} i x^{n_1} y^{n_2} & \text{for } n_1, n_2 \text{ odd.} \end{cases}$$

Similarly, if $n_1 + n_2$ is odd, then

$$\begin{aligned} \Delta_{\underline{X}}^{k+\ell+\frac{m-2}{2}} \left[(u(r, \rho) + \underline{\omega} \underline{\nu} v(r, \rho)) \underline{x}^{n_1} \underline{y}^{n_2} P_{k-n_1}(\underline{x}) P_{\ell-n_2}(\underline{y}) \right] \\ = \text{Ft}_{p,q}^{n_1+n_2,-} [w(\underline{z}) h^-(x, y), P_{k-n_1}(\underline{x}), P_{\ell-n_2}(\underline{y})] (\underline{X}), \end{aligned}$$

$$\text{where } h^-(x, y) = \begin{cases} (-1)^{\frac{n_1+n_2-1}{2}} x^{n_1} y^{n_2} & \text{for } n_1 \text{ odd, } n_2 \text{ even} \\ (-1)^{\frac{n_1+n_2-1}{2}} i x^{n_1} y^{n_2} & \text{for } n_1 \text{ even, } n_2 \text{ odd.} \end{cases}$$

Clearly, h^\pm satisfies $\partial_z^{n_1+n_2+1} h^\pm = 0$ and for that reason

$$\partial_z^{n_1+n_2+1} (w(\underline{z}) h^\pm(x, y)) = w(\underline{z}) \partial_z^{n_1+n_2+1} h^\pm(x, y) = 0.$$

Consequently, the functions $w(\underline{z}) h^\pm(x, y)$ are solutions of (2) for $\mu = n_1 + n_2$. The result now follows from Theorem 3. \square

We conclude with some examples involving the homogeneous polynomials $H_k(\underline{x}) = \langle \underline{x}, \underline{t} \rangle^k$, $H_\ell(\underline{y}) = \langle \underline{y}, \underline{s} \rangle^\ell$, where $\underline{t} \in \mathbb{R}^p$ and $\underline{s} \in \mathbb{R}^q$ are arbitrary fixed vectors. In order to avoid too long computations we have chosen the cases $p = q = 3$, $k = 1, 2$ and $\ell = 1$.

$$\begin{aligned} \text{Ft}_{3,3}^+ [\underline{z}^5, \langle \underline{x}, \underline{t} \rangle, \langle \underline{y}, \underline{s} \rangle] (\underline{X}) &= \frac{10}{r^3} \langle \underline{x}, \underline{t} \rangle \langle \underline{y}, \underline{s} \rangle + \frac{6 \underline{x} \underline{y}}{r^5} \langle \underline{x}, \underline{t} \rangle \langle \underline{y}, \underline{s} \rangle - \frac{2 \underline{t} \underline{y}}{r^3} \langle \underline{y}, \underline{s} \rangle \\ &\quad + \frac{(5r^2 + 3\rho^2) \underline{x} \underline{s}}{r^5} \langle \underline{x}, \underline{t} \rangle - \frac{(5r^2 + \rho^2) \underline{t} \underline{s}}{r^3} \end{aligned}$$

$$\text{Ft}_{3,3}^+ [\underline{z}^8, \langle \underline{x}, \underline{t} \rangle, \langle \underline{y}, \underline{s} \rangle] (\underline{X}) = 10 \langle \underline{x}, \underline{t} \rangle \langle \underline{y}, \underline{s} \rangle - 2 \underline{t} \underline{y} \langle \underline{y}, \underline{s} \rangle + 2 \underline{x} \underline{s} \langle \underline{x}, \underline{t} \rangle + (r^2 - \rho^2) \underline{t} \underline{s}$$

$$\begin{aligned} \text{Ft}_{3,3}^+ [\underline{z}^{10}, \langle \underline{x}, \underline{t} \rangle, \langle \underline{y}, \underline{s} \rangle] (\underline{X}) &= 140(r^2 - \rho^2) \langle \underline{x}, \underline{t} \rangle \langle \underline{y}, \underline{s} \rangle - 56 \underline{x} \underline{y} \langle \underline{x}, \underline{t} \rangle \langle \underline{y}, \underline{s} \rangle \\ &\quad - 4(7r^2 - 5\rho^2) \underline{t} \underline{y} \langle \underline{y}, \underline{s} \rangle + 4(5r^2 - 7\rho^2) \underline{x} \underline{s} \langle \underline{x}, \underline{t} \rangle + (5r^4 - 14r^2 \rho^2 + 5\rho^4) \underline{t} \underline{s} \end{aligned}$$

$$\begin{aligned} \text{Ft}_{3,3}^- [i \underline{z}^6, \langle \underline{x}, \underline{t} \rangle, \langle \underline{y}, \underline{s} \rangle] (\underline{X}) &= \frac{2 \underline{x}}{\rho^3} \langle \underline{x}, \underline{t} \rangle \langle \underline{y}, \underline{s} \rangle - \frac{(3r^2 + 5\rho^2) \underline{y}}{\rho^5} \langle \underline{x}, \underline{t} \rangle \langle \underline{y}, \underline{s} \rangle \\ &\quad + \frac{(r^2 + 5\rho^2)}{\rho^3} (\underline{t} \langle \underline{y}, \underline{s} \rangle + \underline{s} \langle \underline{x}, \underline{t} \rangle) \end{aligned}$$

$$\text{Ft}_{3,3}^- [\underline{z}^9, \langle \underline{x}, \underline{t} \rangle, \langle \underline{y}, \underline{s} \rangle] (\underline{X}) = 2(\underline{x} - \underline{y}) \langle \underline{x}, \underline{t} \rangle \langle \underline{y}, \underline{s} \rangle + (r^2 - \rho^2) (\underline{t} \langle \underline{y}, \underline{s} \rangle + \underline{s} \langle \underline{x}, \underline{t} \rangle)$$

$$\begin{aligned} \text{Ft}_{3,3}^- [\underline{z}^{11}, \langle \underline{x}, \underline{t} \rangle^2, \langle \underline{y}, \underline{s} \rangle] (\underline{X}) &= 8(5\underline{x} - 7\underline{y}) \langle \underline{x}, \underline{t} \rangle^2 \langle \underline{y}, \underline{s} \rangle - 4(7r^2 - 5\rho^2) |\underline{t}|^2 \underline{y} \langle \underline{y}, \underline{s} \rangle \\ &\quad + 4(5r^2 - 7\rho^2) (2\underline{t} \langle \underline{x}, \underline{t} \rangle \langle \underline{y}, \underline{s} \rangle + |\underline{t}|^2 \underline{x} \langle \underline{y}, \underline{s} \rangle + \underline{s} \langle \underline{x}, \underline{t} \rangle^2) + (5r^4 - 14r^2 \rho^2 + 5\rho^4) |\underline{t}|^2 \underline{s} \end{aligned}$$

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